V. *On the Calculus of Symbols, with Applications to the Theory of Differential Equations.*

By W. H. L. RUSSELL, *Esq., A.B.* *Communicated by* ARTHUR CAYLEY, *Esq., F.R.S.*

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THE calculus of generating functions, discovered by LAPLACE, was, as is well known, highly instrumental in calling the attention of mathematicians to the analogy which exists between differentials and powers. This analogy was perceived at length to involve an essential identity, and several analysts devoted themselves to the improvement of the new methods of calculation which were thus called into existence. For a long time the modes of combination assumed to exist between different classes of symbols were those of ordinary algebra; and this sufficed for investigations respecting functions of differential coefficients and constants, and consequently for the integration of linear differential equations, with constant coefficients. The laws of combination of ordinary algebraical symbols may be divided into the commutative and distributive laws; and the number of symbols in the higher branches of mathematics, which are commutative with respect to one another, is very small. It became then necessary to invent an algebra of non-commutative symbols. This important step was effected by Professor BOOLE, for certain classes of symbols, in his well-known and beautiful memoir published in the Transactions of this Society for the year 1844, and the object of the paper which I have now the honour to lay before the Society is to perfect and develope the methods there employed.

For this purpose I have constructed systems of multiplication and division for functions of non-commutative symbols, subject to the same laws of combination as those assumed in Professor BOOLE's memoir, and I thus arrive at equations of great utility in the integration of linear differential equations with variable coefficients.

I then proceed to develope certain general theorems, which will, I hope, be found interesting. I have applied the methods of multiplication, as just explained, to deduce theorems for non-commutative symbols analogous to the binomial and multinomial theorems of ordinary algebra.

Lastly, I have shown how to employ the equations deduced in the earlier part of this paper in the integration of linear differential equations. I have, for this purpose, made use of methods closely resembling the method of divisors which has so long been used in resolving ordinary algebraical equations. The whole paper will, I hope, be found to be a step upwards in the important subject of which it treats. I shall just observe, that the symbolical combinations used in this paper may also be applied to the calculus of finite differences, as may be seen in Professor BOOLE's memoir.

SECTION I. *On the Principles of Symbolical Algebra.*

Let (ϱ) and (π) be two functional symbols combining according to the law $\varrho^n f(\pi)u =f(\pi-n)\varrho^n u$, where (u) is the subject. We shall suppose throughout this paper that $\varrho=x,\ \pi=x\frac{d}{dx}$.

Let P, Q, and R be three functions of (π) and (ϱ), such that PQ acting on any subject is equivalent to R acting on the same subject, or PQ=R. We shall say that P externally multiplies Q, and is an external factor of R. In like manner we shall say that Q internally multiplies P, and is an internal factor of R. We shall also say that R is externally divisible by P, internally by Q.

We easily see the truth of the following symbolical equations depending on the laws of combination assumed above:—

$$\begin{aligned}(\varrho^{\alpha}\pi^{\beta})(\varrho^{a}\pi^{b})u\quad &=\varrho^{\alpha+a}(\pi+a)^{\beta}\pi^{b}u\\ (\varrho^{\alpha}\pi^{\beta})(\varrho^{a}\pi^{b})^{-1}u&=\varrho^{\alpha-a}(\pi-a)^{\beta-b}u.\end{aligned}$$

We shall commence with instances of symbolical multiplication and division, when the multipliers and divisors are monomials.

The following is an instance of external multiplication:—

$$\varrho\pi(\varrho^2+\varrho\pi+\pi^2)=\varrho^3(\pi+2)+\varrho^2(\pi^2+\pi)+\varrho\pi^3;$$

the following is an instance of internal multiplication:—

$$(2\varrho^2-3\varrho\pi^2+(\pi^2+\pi))\varrho\pi^2=2\varrho^3\pi^2-3\varrho^2(\pi^2+\pi)^2+\varrho\pi^2(\pi+1)(\pi+2);$$

the following of external division:—

$$(\varrho^2\pi)^{-1}(\varrho^4(\pi+2)-2\varrho^3\pi(\pi+1)+3\varrho^2\pi^3)=\varrho^2-2\varrho\pi+3\pi^2;$$

the following of internal division:—

$$(\varrho^3\pi+3\varrho^2(\pi^2+\pi)+\varrho\pi(\pi+1)^2)(\varrho\pi)^{-1}=\varrho^2+3\varrho\pi+\pi^2.$$

We shall now consider cases where the multipliers and divisors are polynomials.

The following are instances of external multiplication:—

$$\begin{array}{l}
\varrho+\pi\\
\underline{\varrho-\pi\quad}\\
\varrho^2+\varrho\pi\\
\underline{\quad-\varrho(\pi+1)-\pi^2}\\
\varrho^2\quad-\quad\varrho\quad-\quad\pi^2
\end{array}$$

$$\begin{array}{l}
\varrho\pi^2-(\pi+1)\\
\underline{\varrho\pi-\pi^2\qquad\qquad\qquad}\\
\varrho^2(\pi+1)\pi^2-\varrho\pi(\pi+1)\\
\underline{\qquad\qquad-\varrho(\pi+1)^2\pi^2+\pi^2(\pi+1)}\\
\varrho^2(\pi+1)\pi^2-\varrho\pi(\pi+1)(\pi^2+\pi+1)+\pi^2(\pi+1).
\end{array}$$

The following are instances of internal multiplication:—

$$\begin{array}{l}
\varrho+\pi \\
\underline{\varrho-\pi} \\
\varrho^2+\varrho(\pi+1) \\
\underline{\quad-\varrho\pi \qquad\quad -\pi^2} \\
\varrho^2+\varrho \qquad\quad -\pi^2
\end{array}$$

$$\begin{array}{l}
\varrho\pi^2-(\pi+1) \\
\underline{\varrho\pi - \pi^2 \qquad\qquad\qquad} \\
\varrho^2(\pi+1)^2\pi^2-\varrho\pi(\pi+2) \\
\underline{\qquad\qquad -\varrho\pi^4 \qquad\qquad\quad +\pi^2(\pi+1)} \\
\varrho^2(\pi+1)^2\pi^2-\varrho\pi(\pi^3+\pi+2)+\pi^2(\pi+1).
\end{array}$$

The results of the four last examples may be written thus:—

$$(\varrho-\pi)(\varrho+\pi)=\varrho^2-\varrho-\pi^2$$

$$(\varrho\pi-\pi^2)(\varrho\pi^2-(\pi+1))=(\varrho^2\pi-\varrho(\pi^2+\pi+1)+\pi)\pi(\pi+1)$$

$$(\varrho+\pi)(\varrho-\pi)=\varrho^2+\varrho-\pi^2$$

$$(\varrho\pi^2-(\pi+1))(\varrho\pi-\pi^2)=(\varrho^2(\pi^2+\pi)-\varrho(\pi^2-\pi+2)+\pi)\pi(\pi+1).$$

I shall now give some examples of external division, the divisor being a polynomial.

$$\begin{array}{l}
\varrho+\pi\big)\varrho^3+2\varrho^2(\pi+1)+\varrho\pi(2\pi+1)+\pi^3\big(\varrho^2+\varrho\pi+\pi^2 \\
\qquad\underline{\varrho^2+\ \varrho^2(\pi+2)} \\
\qquad\qquad\varrho^2\pi+\varrho\pi(2\pi+1) \\
\qquad\qquad\underline{\varrho^2\pi+\varrho\pi(\ \pi+1)} \\
\qquad\qquad\qquad\varrho\pi^2+\pi^3 \\
\qquad\qquad\qquad\underline{\varrho\pi^2+\pi^3}
\end{array}$$

$$\begin{array}{l}
\varrho\pi+\pi^2\big)\varrho^3(\pi+2)+\varrho^2(2\pi+3)-\varrho(3\pi^2+3\pi+1)+\pi^4\big(\varrho^2-\varrho(\pi+1)+\pi^2 \\
\qquad\underline{\varrho^3(\pi+2)+\varrho^2(\ \pi+2)^2} \\
\qquad\qquad-\varrho^2(\ \pi+1)^2-\varrho(3\pi^2+3\pi+1) \\
\qquad\qquad\underline{-\varrho^2(\ \pi+1)^2-\varrho(\ \pi+1)^3 \qquad\quad} \\
\qquad\qquad\qquad\varrho\pi^3+\pi^4 \\
\qquad\qquad\qquad\underline{\varrho\pi^3+\pi^4}
\end{array}$$

We shall next consider some examples of internal division.

$$\begin{array}{l}
\varrho+\pi\big)\varrho^3+\varrho^2(2\pi+1)+\varrho(2\pi^2+2\pi+1)+\pi^3\big(\varrho^2+\varrho\pi+\pi^2 \\
\qquad\underline{\varrho^3+\varrho^2\pi} \\
\qquad\qquad\varrho^2(\pi+1)+\varrho(2\pi^2+2\pi+1) \\
\qquad\qquad\underline{\varrho^2(\pi+1)+\varrho\pi^2} \\
\qquad\qquad\qquad\qquad\varrho(\pi+1)^2+\pi^3 \\
\qquad\qquad\qquad\qquad\underline{\varrho(\pi+1)^2+\pi^3}
\end{array}$$

$$\begin{array}{l}
\varrho\pi+\pi^2\big)\varrho^3\pi-2\varrho^2\pi+\varrho(\pi^2+\pi)+\pi^4\big(\varrho^2-\varrho(\pi+1)+\pi^2\\
\qquad\qquad\underline{\varrho^3\pi+\varrho^2\pi^2}\\
\qquad\qquad\qquad-\varrho^2(\pi^2+2\pi)+\varrho(\pi^2+\pi)\\
\qquad\qquad\qquad\underline{-\varrho^2(\pi^2+2\pi)-\varrho(\pi+1)\pi^2}\\
\qquad\qquad\qquad\qquad\qquad\qquad\varrho\pi(\pi+1)^2+\pi^4\\
\qquad\qquad\qquad\qquad\qquad\qquad\underline{\varrho\pi(\pi+1)^2+\pi^4}
\end{array}$$

The results of the four last examples may be written thus:

$$(\varrho+\pi)^{-1}(\varrho^3+2\varrho^2(\pi+1)+\varrho\pi(2\pi+1)+\pi^3)=\varrho^2+\varrho\pi+\pi^2$$

$$(\varrho\pi+\pi^2)^{-1}(\varrho^3(\pi+2)+\varrho^2(2\pi+3)-\varrho(3\pi^2+3\pi+1)+\pi^4)=\varrho^2-\varrho(\pi+1)+\pi^2$$

$$(\varrho^3+\varrho^2(2\pi+1)+\varrho(2\pi^2+2\pi+1)+\pi^3)(\varrho+\pi)^{-1}=\varrho^2+\varrho\pi+\pi^2$$

$$(\varrho^3\pi-2\varrho^2\pi+\varrho(\pi^2+\pi)+\pi^4)(\varrho\pi+\pi^2)^{-1}=\varrho^2-\varrho(\pi+1)+\pi^2.$$

I now come to two propositions of great importance.

First, to determine the condition that $\varrho\psi_1(\pi)+\psi_0(\pi)$ shall divide the symbolical function

$$\varrho^n\phi_n(\pi)+\varrho^{n-1}\phi_{n-1}(\pi)+\varrho^{n-2}\phi_{n-2}(\pi)+\&\text{c.}+\varrho\phi_1(\pi)+\phi_0(\pi)$$

internally without a remainder,

$$\begin{array}{l}
\varrho\psi_1(\pi)+\psi_0(\pi)\big)\varrho^n\phi_n(\pi)+\varrho^{n-1}\phi_{n-1}(\pi)+\varrho^{n-2}\phi_{n-2}(\pi)+\ldots+\varrho^2\phi_2(\pi)+\varrho\phi_1(\pi)+\phi_0(\pi)\\
\qquad\underline{\varrho^n\phi_n(\pi)+\varrho^{n-1}\frac{\psi_0(\pi)}{\psi_1(\pi-1)}\phi_n(\pi-1)\qquad\qquad}\\
\qquad\qquad\qquad\varrho^{n-1}\left\{\phi_{n-1}(\pi)-\frac{\psi_0(\pi)}{\psi_1(\pi-1)}\phi_n(\pi-1)\right\}+\varrho^{n-2}\phi_{n-2}(\pi)\\
\qquad\qquad\qquad\underline{\varrho^{n-1}\left\{\phi_{n-1}(\pi)-\frac{\psi_0(\pi)}{\psi_1(\pi-1)}\phi_n(\pi-1)\right\}+\varrho^{n-2}\left\{\frac{\psi_0(\pi)}{\psi_1(\pi-1)}\phi_{n-1}(\pi-1)-\frac{\psi_0(\pi)\psi_0(\pi-1)}{\psi_1(\pi-1)\psi_1(\pi-2)}\phi_n(\pi-2)\right\}},\\
\qquad\qquad\qquad\qquad\qquad\varrho^{n-2}\left\{\phi_{n-2}(\pi)-\frac{\psi_0(\pi)}{\psi_1(\pi-1)}\phi_{n-1}(\pi-1)+\frac{\psi_0(\pi)\psi_0(\pi-1)}{\psi_1(\pi-1)\psi_1(\pi-2)}\phi_n(\pi-2)\right\}+\&\text{c.}
\end{array}$$

where the symbolical quotient is

$$\varrho^{n-1}\frac{\phi_n(\pi-1)}{\psi_1(\pi-1)}+\varrho^{n-2}\left\{\frac{\phi_{n-1}(\pi-1)}{\psi_1(\pi-1)}-\frac{\psi_0(\pi-1)}{\psi_1(\pi-1)\psi_1(\pi-2)}\phi_n(\pi-2)\right\}+\varrho^{n-3}\left\{\frac{\phi_{n-2}(\pi-1)}{\psi_1(\pi-1)}\right.$$
$$\left.-\frac{\psi_0(\pi-1)}{\psi_1(\pi-1)\psi_1(\pi-2)}\phi_{n-1}(\pi-2)+\frac{\psi_0(\pi-1)\psi_0(\pi-2)}{\psi_1(\pi-1)\psi_1(\pi-2)\psi_1(\pi-3)}\phi_n(\pi-3)\right\}+\&\text{c.}$$

The required condition is found by equating the remainder to zero; and we have

$$\phi_0(\pi)-\frac{\psi_0(\pi)}{\psi_1(\pi-1)}\phi_1(\pi-1)+\frac{\psi_0(\pi)\psi_0(\pi-1)}{\psi_1(\pi-1)\psi_1(\pi-2)}\phi_2(\pi-2)-\frac{\psi_0(\pi)\psi_0(\pi-1)\psi_0(\pi-2)}{\psi_1(\pi-1)\psi_1(\pi-2)\psi_1(\pi-3)}\phi_3(\pi-3)+\&\text{c.}$$
$$\pm\frac{\psi_0\pi\psi_0(\pi-1)\psi_0(\pi-2)\ldots\psi_0(\pi-n+1)}{\psi_1(\pi-1)\psi_1(\pi-2)\psi_1(\pi-3)\ldots\psi_1(\pi-n)}\phi_n(\pi-n)=0,$$

where $\varrho\psi_1(\pi)+\psi_0(\pi)$ is an internal factor of $\varrho^n\phi_n(\pi)+\varrho^{n-1}\phi_{n-1}(\pi)+$ &c. $+\phi_0(\pi)$.

Hence we see how we may resolve the symbolical function

$$\varrho^n\phi_n(\pi)+\varrho^{n-1}\phi_{n-1}(\pi)+\varrho^{n-2}\phi_{n-2}(\pi)+\ldots\ldots+\varrho\phi_1(\pi)+\phi_0(\pi)$$

into factors in all possible cases.

Put

$$\psi_0(\pi)=A_0+B_0\pi+C_0\pi^2+\&c.,$$
$$\psi_1(\pi)=A_1+B_1\pi+C_1\pi^2+\&c.,$$

and substitute in the above equation, and equate the resulting coefficients of π to zero. We shall thus be furnished with equations for determining the values of A_0, B_0, &c., A_1, B_1, &c. in all cases in which the above symbolical function is capable of resolution. We thus obtain the values of $\psi_0(\pi)$, $\psi_1(\pi)$, and of the symbolical quotient. We next ascertain if the symbolical quotient admits of an internal factor, and repeating the process we at length resolve the above symbolical function into factors of the form

$$(\varrho\psi_1^{(n)}(\pi)+\psi_0^{(n)}(\pi))(\varrho\psi_1^{(n-1)}(\pi)+\psi_0^{(n-1)}(\pi))\ldots(\varrho\psi_1(\pi)+\psi_0(\pi)).$$

To determine the condition that $\varrho\psi_1(\pi)+\psi_0(\pi)$ shall divide the symbolical function

$$\varrho^n\phi_n(\pi)+\varrho^{n-1}\phi_{n-1}(\pi)+\varrho^{n-2}\phi_{n-2}(\pi)+\&c.+\varrho\phi_1(\pi)+\phi_0(\pi)$$

externally without a remainder,

$$\varrho\psi_1(\pi)+\psi_0(\pi))\varrho^n\phi_n(\pi)+\varrho^{n-1}\phi_{n-1}(\pi)+\varrho^{n-2}\phi_{n-2}(\pi)+\&c.+\varrho^2\phi_2(\pi)+\varrho\phi_1(\pi)+\phi_0(\pi)$$

$$\underline{\varrho^n\phi_n(\pi)+\varrho^{n-1}\frac{\psi_0(\pi+n-1)}{\psi_1(\pi+n-1)}\phi_n(\pi)}$$

$$\varrho^{n-1}\left\{\phi_{n-1}(\pi)-\frac{\psi_0(\pi+n-1)}{\psi_1(\pi+n-1)}\phi_n(\pi)\right\}+\varrho^{n-2}\phi_{n-2}(\pi)$$

$$\underline{\varrho^{n-1}\left\{\phi_{n-1}(\pi)-\frac{\psi_0(\pi+n-1)}{\psi_1(\pi+n-1)}\phi_n(\pi)\right\}+\varrho^{n-2}\left\{\frac{\psi_0(\pi+n-2)}{\psi_1(\pi+n-2)}\phi_{n-1}(\pi)-\frac{\psi_0(\pi+n-2)\psi_0(\pi+n-1)}{\psi_1(\pi+n-2)\psi_1(\pi+n-1)}\phi_n(\pi)\right\}}$$

$$\varrho^{n-2}\left\{\phi_{n-2}(\pi)-\frac{\psi_0(\pi+n-2)}{\psi_1(\pi+n-2)}\phi_{n-1}(\pi)+\frac{\psi_0(\pi+n-2)\psi_0(\pi+n-1)}{\psi_1(\pi+n-2)\psi_1(\pi+n-1)}\phi_n(\pi)\right\}+\&c.$$

where the symbolical quotient is

$$\varrho^{n-1}\cdot\frac{\phi_n(\pi)}{\psi_1(\pi+n-1)}+\varrho^{n-2}\left\{\frac{\phi_{n-1}(\pi)}{\psi_1(\pi+n-2)}-\frac{\psi_0(\pi+n-1)}{\psi_1(\pi+n-2)\psi_1(\pi+n-1)}\phi_n(\pi)\right\}$$
$$+\varrho^{n-3}\left\{\frac{\phi_{n-2}\pi}{\psi_1(\pi+n-3)}-\frac{\psi_0(\pi+n-2)}{\psi_1(\pi+n-3)\psi_1(\pi+n-2)}\phi_{n-1}(\pi)+\frac{\psi_0(\pi+n-2)\psi_0(\pi+n-1)}{\psi_1(\pi+n-3)\psi_1(\pi+n-2)\psi_1(\pi+n-1)}\phi_n(\pi)\right\}+\&c.$$

The required condition is found by equating the remainder to zero: whence we have

$$\phi_0(\pi)-\frac{\psi_0(\pi)}{\psi_1(\pi)}\phi_1(\pi)+\frac{\psi_0(\pi)\psi_0(\pi+1)}{\psi_1(\pi)\psi_1(\pi+1)}\phi_2(\pi)-\frac{\psi_0(\pi)\psi_0(\pi+1)\psi_0(\pi+2)}{\psi_1(\pi)\psi_1(\pi+1)\psi_1(\pi+2)}\phi_3(\pi)+\&c.$$
$$\pm\frac{\psi_0(\pi)\psi_0(\pi+1)\psi_0(\pi+2)\ldots\psi_0(\pi+n-1)}{\psi_1(\pi)\psi_1(\pi+1)\psi_1(\pi+2)\ldots\psi_1(\pi+n-1)}\phi_n(\pi)=0.$$

In the next investigation we shall suppose the symbolical function arranged in powers of (π) instead of powers of (ϱ). To determine the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of the symbolical function

$$\phi_3(\varrho)\pi^3+\phi_2(\varrho)\pi^2+\phi_1(\varrho)\pi+\phi_0(\varrho).$$

We easily see that

$$\pi\varphi(\varrho)=\varphi(\varrho)\pi+\varrho\frac{d}{d\varrho}\varphi(\varrho)$$

$$\pi^2\varphi(\varrho)=\varphi(\varrho)\pi^2+2\varrho\frac{d}{d\varrho}\varphi(\varrho)\pi+\left(\varrho\frac{d}{d\varrho}\right)^2\varphi(\varrho).$$

Hence we shall have

$$\varphi^3(\varrho)\pi^3\{\psi_1\varrho\,.\,\pi\}^{-1}=\varphi_3(\varrho)\pi^2\,.\,\frac{1}{\psi_1\varrho}=\frac{\varphi_3\varrho}{\psi_1\varrho}\pi^2+2\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1\varrho}\cdot\pi+\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)^2\frac{1}{\psi_1(\varrho)};$$

$$\begin{aligned}\therefore\ \varphi_3(\varrho)\,.\,\pi^3&=\left\{\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\pi^2+2\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1\varrho}\pi+\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)^2\cdot\frac{1}{\psi_1\varrho}\right\}\{\psi_1(\varrho)\,.\,\pi+\psi_0(\varrho)-\psi_0(\varrho)\}\\
&=\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\pi^2\{\psi_1(\varrho)\,.\,\pi+\psi_0(\varrho)\}+2\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1\varrho}\pi\psi_1(\varrho)\pi\\
&\quad+\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)^2\frac{1}{\psi_1\varrho}\cdot\psi_1(\varrho)\pi-\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\pi^2\psi_0(\varrho)\\
&=\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\pi^2\{\psi_1(\varrho)\pi+\psi_0(\varrho)\}+2\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1(\varrho)}\left\{\psi_1(\varrho)\pi+\varrho\frac{d}{d\varrho}\psi_1(\varrho)\right\}\pi\\
&\qquad+\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)^2\cdot\frac{1}{\psi_1(\varrho)}\cdot\psi_1(\varrho)\,.\,\pi\\
&\quad-\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\left\{\psi_0(\varrho)\,.\,\pi^2+2\left(\varrho\frac{d}{d\varrho}\right)\psi_0(\varrho)\,.\,\pi+\left(\varrho\frac{d}{d\varrho}\right)^2\psi_0(\varrho)\right\};\end{aligned}$$

$$\begin{aligned}\therefore\ &\varphi_3(\varrho)\,.\,\pi^3+\varphi_2(\varrho)\pi^2+\varphi_1(\varrho)\,.\,\pi+\varphi_0(\varrho)\\
&=\frac{\varphi_3\varrho}{\psi_1\varrho}\pi^2\{\psi_1)(\varrho)\pi+\psi_0(\varrho)\}\\
&+\left\{\varphi_2(\varrho)+2\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1(\varrho)}\cdot\psi_1(\varrho)-\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\psi_0(\varrho)\right\}\pi^2\\
&+\left\{\varphi_1(\varrho)+2\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1(\varrho)}\varrho\frac{d}{d\varrho}\psi_1(\varrho)+\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)^2\frac{1}{\psi_1(\varrho)}\cdot\psi_1(\varrho)-2\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\left(\varrho\frac{d}{d\varrho}\right)\psi_0(\varrho)\right\}\pi\\
&+\varphi_0(\varrho)-\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\left(\varrho\frac{d}{d\varrho}\right)^2\psi_0(\varrho);\end{aligned}$$

where we may put

$$\begin{aligned}\theta_2(\varrho)&=\varphi_2(\varrho)+2\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1(\varrho)}\psi_1(\varrho)-\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\psi_0(\varrho)\\
\theta_1\varrho&=\varphi_1(\varrho)+2\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1(\varrho)}\left(\varrho\frac{d}{d\varrho}\right)\psi_1(\varrho)+\varphi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)^2\frac{1}{\psi_1(\varrho)}\cdot\psi_1(\varrho)-2\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\left(\varrho\frac{d}{d\varrho}\right)\psi_0(\varrho)\\
\theta_0\varrho&=\varphi_0(\varrho)-\frac{\varphi_3(\varrho)}{\psi_1(\varrho)}\left(\varrho\frac{d}{d\varrho}\right)^2\psi_0(\varrho)\end{aligned}$$

$$\theta_2(\varrho)\pi^2\{\psi_1(\varrho)\,.\,\pi\}^{-1}=\theta_2(\varrho)\,.\,\pi\,.\,\frac{1}{\psi_1(\varrho)}=\frac{\theta_2(\varrho)}{\psi_1(\varrho)}\pi+\theta_2(\varrho)\,.\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1(\varrho)};$$

$$\begin{aligned}\theta_2(\varrho)\pi^2&=\frac{\theta_2(\varrho)}{\psi_1(\varrho)}\pi\{\psi_1(\varrho)\pi+\psi_0(\varrho)\}\\
&\quad-\frac{\theta_2(\varrho)}{\psi_1(\varrho)}\left\{\psi_0(\varrho)\,.\,\pi+\varrho\frac{d}{d\varrho}\psi_0(\varrho)\right\}+\theta_2(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1(\varrho)}\cdot\psi_1(\varrho)\pi\\
&\quad=\frac{\theta_2\varrho}{\psi_1\varrho}\pi\{\psi_1(\varrho)\pi+\psi_0(\varrho)\}-\theta_2(\varrho)\left\{\frac{\psi_0(\varrho)}{\psi_1(\varrho)}-\varrho\frac{d}{d\varrho}\left(\frac{1}{\psi_1\varrho}\right)\psi_1(\varrho)\right\}\pi-\frac{\theta_2\varrho}{\psi_1\varrho}\varrho\frac{d}{d\varrho}\psi_0(\varrho).\end{aligned}$$

Then

$$\theta_2(\varrho)\pi^2+\theta_1(\varrho)\pi+\theta_0(\varrho)=\frac{\theta_2(\varrho)}{\psi_1(\varrho)}\pi\{\psi_1(\varrho)\pi+\psi_0(\varrho)\}$$

$$+\left\{\theta_1(\varrho)-\frac{\theta_2\varrho\psi_0\varrho}{\psi_1\varrho}+\theta_2\varrho\,\varrho\frac{d}{d\varrho}\cdot\frac{1}{\psi_1\varrho}\cdot\psi_1\varrho\right\}\pi$$

$$+\left\{\theta_0(\varrho)-\frac{\theta_2(\varrho)}{\psi_1(\varrho)}\varrho\frac{d}{d\varrho}\psi_0\varrho\right\}.$$

Put, for the sake of simplicity,

$$\omega_1(\varrho)=\theta_1(\varrho)-\frac{\theta_2(\varrho)\psi_0\varrho}{\psi_1\varrho}+\theta_2\varrho\,.\,\varrho\frac{d}{d\varrho}\cdot\frac{1}{\psi_1\varrho}\cdot\psi_1\varrho$$

$$\omega_0(\varrho)=\theta_0\varrho-\frac{\theta_2\varrho}{\psi_1\varrho}\varrho\frac{d}{d\varrho}\psi_0(\varrho)$$

$$\omega_1(\varrho)\,.\,\pi\,.\,\{\psi_1\varrho\pi\}^{-1}=\frac{\omega_1\varrho}{\psi_1\varrho},$$

$$\therefore\ \omega_1(\varrho)\pi=\frac{\omega_1(\varrho)}{\psi_1(\varrho)}\{\psi_1\varrho\,.\,\pi+\psi_0\varrho\}-\frac{\omega_1(\varrho)}{\psi_1\varrho}\psi_0(\varrho),$$

$$\omega_1(\varrho)\pi+\omega_0(\varrho)=\frac{\omega_1\varrho}{\psi_1\varrho}(\psi_1\varrho\,.\,\pi+\psi_0\varrho)+\omega_0(\varrho)-\frac{\omega_1(\varrho)}{\psi_1(\varrho)}\psi_0(\varrho).$$

Hence the condition that $\psi_1(\varrho)\pi+\psi_0\varrho$ may be an internal factor of

$$\phi_3(\varrho)\pi^3+\phi_1(\varrho)\pi^2+\phi_1(\varrho)\pi+\phi_0(\varrho)$$

is equivalent to the equation

$$\omega_1(\varrho)\psi_0(\varrho)-\omega_0(\varrho)\psi_1(\varrho)=0.$$

Hence, substituting for $\omega_1(\varrho)$ and $\omega_0(\varrho)$ their values, we have

$$\theta_1(\varrho)\psi_0(\varrho)-\frac{\theta_2\varrho}{\psi_1\varrho}(\psi_0(\varrho))^2+\theta_2(\varrho)\psi_0(\varrho)\varrho\frac{d}{d\varrho}\,\frac{1}{\psi_1\varrho}\cdot\psi_1\varrho-\theta_0(\varrho)\psi_1(\varrho)+\theta_2(\varrho)\varrho\frac{d}{d\varrho}\psi_0(\varrho)=0\,;$$

or, again substituting for $\theta_0(\varrho)$, $\theta_1(\varrho)$, $\theta_2(\varrho)$, we have

$$\left\{\phi_2(\varrho)+2\phi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1\varrho}\cdot\psi_1\varrho-\frac{\phi_3(\varrho)}{\psi_1(\varrho)}\psi_0(\varrho)\right\}\left\{\varrho\frac{d}{d\varrho}\psi_0\varrho+\psi_0(\varrho)\varrho\frac{d}{d\varrho}\cdot\frac{1}{\psi_1\varrho}\psi_1(\varrho)-\frac{(\psi_0\varrho)^2}{\psi_1\varrho}\right\}$$

$$+\psi_0\varrho\left\{\phi_1(\varrho)+2\phi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\frac{1}{\psi_1\varrho}\left(\varrho\frac{d}{d\varrho}\right)\psi_1\varrho+\phi_3(\varrho)\left(\varrho\frac{d}{d\varrho}\right)^2\frac{1}{\psi_1\varrho}\cdot\psi_1\varrho-2\frac{\phi_3\varrho}{\psi_1\varrho}\left(\varrho\frac{d}{d\varrho}\right)\psi_0\varrho\right\}$$

$$-\psi_1\varrho\left\{\phi_0(\varrho)-\frac{\phi_3\varrho}{\psi_1(\varrho)}\left(\varrho\frac{d}{d\varrho}\right)^2\psi_0(\varrho)\right\}=0.$$

Had we wished to ascertain the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of

$$\phi_4(\varrho)\pi^4+\phi_3(\varrho)\pi^3+\phi_2(\varrho)\pi^2+\phi_1(\varrho)\pi+\phi_0(\varrho),$$

we must have calculated the value of $\pi^3\phi(\varrho)$. It is evident that for every increase in the degree of the highest power of (π), the labour of the investigation becomes immensely greater, and the result far more complicated. It is, however, of considerable utility in the integration of differential equations, and we shall refer to it again at the close of this paper.

SECTION II. *On some General Theorems.*

I shall now give some theorems in general differentiation and expansion.

Since

$$\left(\frac{1}{\varrho^2}\cdot\pi\right)^n=\left(\frac{1}{\varrho^2}\cdot\pi\right)\left(\frac{1}{\varrho^2}\pi\right)\left(\frac{1}{\varrho^2}\cdot\pi\right)\ldots$$

to n factors, we have

$$\left(\frac{1}{\varrho^2}\cdot\pi\right)^n=\frac{1}{\varrho^{2n}}\,\pi(\pi-2)(\pi-4)\ldots.(\pi-2n+2),$$

$$\therefore\ \left(\frac{1}{\varrho^2}\cdot\pi\right)^n\cdot\frac{1}{\varrho}=\frac{1}{\varrho^{2n+1}}(\pi-1)(\pi-3)\ldots(\pi-2n+1);$$

whence we easily see that

$$\pi(\pi-1)(\pi-2)(\pi-3)\ldots(\pi-2n+1)=\varrho^{2n+1}\left(\frac{1}{\varrho^2}\pi\right)^n\varrho^{2n-1}\left(\frac{1}{\varrho^2}\pi\right)^n,$$

$$\frac{1}{\varrho^{2n}}\pi(\pi-1)(\pi-2)(\pi-3)\ldots(\pi-2n+1)=\varrho\left(\frac{1}{\varrho^2}\pi\right)^n\varrho^{2n-1}\left(\frac{1}{\varrho^2}\pi\right)^n,$$

$$\therefore\ \left(\frac{1}{\varrho}\cdot\pi\right)^{2n}=\varrho\left(\frac{1}{\varrho^2}\cdot\pi\right)^n\varrho^{2n-1}\left(\frac{1}{\varrho^2}\cdot\pi\right)^n;$$

whence we shall have

$$\frac{d^{2n}u}{dx^{2n}}=x\left(\frac{1}{x}\cdot\frac{d}{dx}\right)^n x^{2n-1}\left(\frac{1}{x}\cdot\frac{d}{dx}\right)^n u.$$

If we equate the coefficients of z^r in $(1+z)^{2n}=(1+z)^n(z+1)^n$, we have

$$\frac{2n(2n-1)(2n-2)\ldots(2n-r+1)}{1.2.3\ldots r}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}$$

$$+\frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3\ldots r-1}\cdot n+\frac{n(n-1)\ldots(n-r+3)}{1.2.3\ldots r-2}\cdot\frac{n(n-1)}{1.2}+\ldots;$$

$$\therefore\ 2\pi(2\pi-1)(2\pi-2)\ldots.(2\pi-r+1)$$

$$=\pi(\pi-1)\ldots(\pi-r+1)+r\pi(\pi-1)(\pi-2)\ldots(\pi-r+2)\pi$$

$$+r\frac{r-1}{2}\pi(\pi-1)(\pi-2)\ldots(\pi-r+3).\pi(\pi-1).$$

Hence, since

$$\left(\frac{1}{\varrho^{\frac{1}{2}}}\pi\right)^r=\frac{1}{\varrho^{\frac{r}{2}}}\pi\left(\pi-\frac{1}{2}\right)(\pi-1)\ldots.\left(\pi-\frac{r-1}{2}\right),$$

we have

$$2^r\varrho^{\frac{r}{2}}\left(\frac{1}{\varrho^{\frac{1}{2}}}\pi\right)^r=\varrho^r\left(\frac{1}{\varrho}\pi\right)^r+r\varrho^{r-1}\left(\frac{1}{\varrho}\pi\right)^{r-1}\varrho\left(\frac{1}{\varrho}\pi\right)$$

$$+r\frac{r-1}{2}\varrho^{r-2}\left(\frac{1}{\varrho}\pi\right)^{r-2}\varrho^2\left(\frac{1}{\varrho}\pi\right)^2+\&c.;$$

whence we find

$$x^{\frac{r}{2}}\left(\frac{d}{d.x^{\frac{1}{2}}}\right)^r u=x^r\frac{d^ru}{du^r}+rx^{r-1}\frac{d^{r-1}}{dx^{r-1}}x\frac{du}{dx}+r\frac{r-1}{2}x^{r-2}\frac{d^{r-2}}{dx^{r-2}}x^2\frac{d^2u}{dx^2}+\&c.^*$$

---

* It has been pointed out to me that this theorem might be more shortly proved by applying VANDERMONDE's theorem to the equation $(2D)^r=(D+D)^r$. I have retained the demonstration in the text merely for the sake of the method.

I now come to the theorems respecting expansion, which I mentioned in the beginning of the paper as analogous to the binomial and multinomial theorems in ordinary algebra.

To expand $(\varrho^2+\varrho\theta(\pi))^n$ in powers of π, where $\theta(\pi)$ is a function of (π), and (n) is a positive integer.

Let us assume

$$(\varrho^2+\varrho\theta(\pi))^n=\varphi_n^{(0)}(\varrho)+\varphi_n^{(1)}(\varrho)\,.\,\pi+\varphi_n^{(2)}(\varrho)\pi^2+\&c.,$$

where

$$\varphi_n^{(0)}(\varrho)=\varrho^{2n}+A_n^{(1)}\varrho^{2n-1}+A_n^{(2)}\varrho^{2n-2}+\&c.$$

$$\varphi_n^{(1)}\varrho\;=B_n^{(0)}\varrho^{2n-1}+B_n^{(1)}\varrho^{2n-2}+B_n^{(2)}\;\varrho^{2n-3}+\&c.\,;$$

$$\therefore\quad \varphi_{n+1}^{(0)}\varrho\;=\varrho^{2n+2}+\quad A_{n+1}^{(1)}\varrho^{2n+1}+\qquad\qquad A_{n+1}^{(2)}\varrho^{2n}+\&c.$$

$$=\varrho^{2n+2}+\quad A_n^{(1)}\varrho^{2n+1}+\qquad\qquad A_n^{(2)}\varrho^{2n}+\&c.$$

$$+\quad\theta(2n)\varrho^{2n+1}+A_n^{(1)}\theta(2n-1)\varrho^{2n}+\&c.\,;$$

$$\therefore\quad A_{n+1}^{(1)}=A_n^{(1)}+\theta(2n),\;\; A_{n+1}^{(2)}=A_n^{(2)}+A_n^{(1)}\theta(2n-1)\,;$$

or

$$A_n^{(1)}=\Sigma\theta(2n),\;\; A_n^{(2)}=\Sigma(\theta(2n-1)\Sigma\theta(2n)).$$

Similarly,

$$A_n^{(3)}=\Sigma\{\theta(2n-2)\Sigma(\theta(2n-1)\Sigma\theta(2n))\}\ldots\&c.$$

Again, we shall have

$$\varphi_{n+1}^{(1)}(\varrho)=B_{n+1}^{(0)}\;\varrho^{2n+1}+\qquad\qquad B_{n+1}^{(1)}\varrho^{2n}+\qquad\qquad B_{n+1}^{(2)}\varrho^{2n-1}+\;\&c.$$

$$=\theta'(2n)\varrho^{2n+1}+A_n^{(1)}\theta'(2n-1)\varrho^{2n}+A_n^{(2)}\theta'(2n-2)\varrho^{2n-1}+\,..$$

$$+B_n^{(0)}\;\;\varrho^{2n+1}+\qquad\qquad B_n^{(1)}\varrho^{2n}+\qquad\qquad B_n^{(2)}\varrho^{2n-1}+\,..$$

$$+B_n^{(0)}\theta(2n-1)\;\varrho^{2n}+B_n^{(1)}\theta(2n-2)\;\varrho^{2n-1}+\;\&c.$$

Consequently

$$B_{n+1}^{(0)}=B_n^{(0)}+\theta'(2n)\,;\qquad\therefore\;\; B_{(n)}^{0}=\Sigma\theta'(2n)$$

$$B_{n+1}^{(1)}=B_n^{(1)}+B_n^{(0)}\theta(2n-1)+A_n^{(1)}\theta'(2n-1)\,;$$

$$\therefore\; B_n^{(1)}\;=\Sigma(\theta(2n-1)\Sigma\theta'(2n))+\Sigma\theta'(2n-1)\Sigma\theta(2n)).$$

Hence we shall have

$$(\varrho^2+\varrho\theta(\pi))^n=\varrho^{2n}+\Sigma\theta(2n)\,.\,\varrho^{2n-1}$$
$$+\Sigma\theta(2n-1)\Sigma\theta(2n)\varrho^{2n-2}+\Sigma\theta(2n-2)\Sigma\theta(2n-1)\Sigma\theta(2n)\varrho^{2n-3}+\;\&c.$$
$$+\{\Sigma\theta'(2n)\varrho^{2n-1}+(\Sigma\theta(2n-1)\Sigma\theta'(2n)+\Sigma\theta'(2n-1)\Sigma\theta(2n))\varrho^{2n-2}+\;\&c.\}\pi+\;\&c.$$

When $\theta(\pi)$ is a rational and entire function of (π),

$$\Sigma\theta(2n),\quad \Sigma(\theta(2n-1)\Sigma(2n))\;\&c.\,;\;\text{and }\Sigma\theta'(2n)\;\&c.$$

can always be obtained in finite terms, as manifestly ought to be the case.

In like manner we shall have

$$\left(\varrho+\frac{1}{\varrho}\theta(\pi)\right)^n=\varrho^n+\Sigma\theta(n)\varrho^{n-2}$$

$$+\Sigma(\theta(n-2)\Sigma\theta(n))\varrho^{n-4}+\Sigma\{\theta(n-4)\Sigma(\theta(n-2)\Sigma\theta(n))\}\varrho^{n-6}+\&c.$$
$$+\{\Sigma\theta'(n)\varrho^{n-2}+(\Sigma\theta'(n-2)\Sigma\theta(n)+\Sigma\theta(n-2)\Sigma\theta'(n))\varrho^{n-4}+\&c.\}\pi+\&c.;$$

and also

$$\begin{aligned}\left(\varrho^2+\frac{1}{\varrho}\theta(\pi)\right)^n&=\varrho^{2n}+\Sigma\theta(2n)\varrho^{2n-3}\\&+\Sigma(\theta(2n-3)\Sigma\theta(2n))\varrho^{2n-6}+\Sigma\{\theta(2n-6)\Sigma(\theta(2n-3)\Sigma\theta(2n))\}\varrho^{2n-9}+\&c.\\&+\{\Sigma\theta'(2n)\varrho^{2n-3}+(\Sigma\theta'(2n-3)\Sigma\theta(2n)+\Sigma\theta(2n-3)\Sigma\theta'(2n))\varrho^{2n-6}+\&c.\}\pi+\&c.\end{aligned}$$

If we put $\theta(\pi)=\pi^2$, it is obvious that the three last theorems will give us the expansions of

$$\left(x^2+x^2\frac{d}{dx}+x^3\frac{d^2}{dx^2}\right)^n,\quad\left(x+\frac{d}{dx}+x\frac{d^2}{dx^2}\right)^n,\quad\text{and of}\quad\left(x^2+\frac{d}{dx}+x\frac{d^2}{dx^2}\right)^n$$

in terms of $x\frac{d}{dx}$.

The same methods of course will apply to all binomials included under the form $(\varrho^\alpha+\varrho^\beta\theta(\pi))^n$. I have found that there is no difficulty in calculating the forms of the coefficients, beyond the labour expended in performing the finite integrations.

To determine that part of the expansion of $(\varrho^\alpha+\varrho^{\alpha-1}\theta_1(\pi)+\varrho^{\alpha-2}\theta_2(\pi)+\varrho^{\alpha-3}\theta_3(\pi)+\&c.)^n$ which is independent of π.

Let us assume

$$(\varrho^\alpha+\varrho^{\alpha-1}\theta_1(\pi)+\varrho^{\alpha-2}\theta_2(\pi)+\varrho^{\alpha-3}\theta_3(\pi)+\ldots)^n=\phi_n^{(0)}(\varrho)+\phi_n^{(1)}(\varrho).\pi+\phi_n^{(2)}(\varrho).\pi^2+\ldots,$$

where

$$\phi_n^{(0)}(\varrho)=\varrho^{\alpha n}+A_n^{(1)}\varrho^{\alpha n-1}+A_n^{(2)}\varrho^{\alpha n-2}+A_n^{(3)}\varrho^{\alpha n-3}+\ldots$$

Then we shall have

$$\begin{aligned}\phi_{n+1}^{(0)}(\varrho)&=\varrho^{\alpha n+\alpha}+A_{n+1}^{(1)}\varrho^{\alpha n+\alpha-1}+A_{n+1}^{(2)}\varrho^{\alpha n+\alpha-2}+A_{n+1}^{(3)}\varrho^{\alpha n+\alpha-3}+\ldots\\&=\varrho^{\alpha n+\alpha}+A_n^{(1)}\varrho^{\alpha n+\alpha-1}+A_n^{(2)}\varrho^{\alpha n+\alpha-2}+A_n^{(3)}\varrho^{\alpha n+\alpha-3}+\ldots\\&\quad+\theta_1(\alpha n)\varrho^{\alpha n+\alpha-1}+A_n^{(1)}\theta_1(\alpha n-1)\varrho^{\alpha n+\alpha-2}+A_n^{(2)}\theta_1(\alpha n-2)\varrho^{\alpha n+\alpha-3}+\ldots\\&\quad+\theta_2(\alpha n)\varrho^{\alpha n+\alpha-2}+A_n^{(1)}\theta_2(\alpha n-1)\varrho^{\alpha n+\alpha-3}+\ldots\\&\quad+\theta_3(\alpha n)\varrho^{\alpha n+\alpha-3}+\ldots\end{aligned}$$

$$\begin{aligned}\therefore\ A_{n+1}^{(1)}&=A_n^{(1)}+\theta_1(\alpha n)\\A_{n+1}^{(2)}&=A_n^{(2)}+A_n^{(1)}\theta_1(\alpha n-1)+\theta_2(\alpha n)\\A_{n+1}^{(3)}&=A_n^{(3)}+A_n^{(2)}\theta_1(\alpha n-2)+A_n^{(1)}\theta_2(\alpha n-1)+\theta_3(\alpha n)\\\&c.&=\&c.;\end{aligned}$$

$$\begin{aligned}\therefore\ A_n^{(1)}&=\Sigma\theta_1(\alpha n)\\A_n^{(2)}&=\Sigma\theta_1(\alpha n-1)\Sigma\theta_1(\alpha n)+\Sigma\theta_2(\alpha n)\\A_n^{(3)}&=\Sigma\theta_1(\alpha n-2)\Sigma\theta_1(\alpha n-1)\Sigma\theta_1(\alpha n)\\&\quad+\Sigma\theta_1(\alpha n-2)\Sigma\theta_2(\alpha n)+\Sigma\theta_2(\alpha n-1)\Sigma\theta_1(\alpha n)+\Sigma\theta_3(\alpha n);\end{aligned}$$

and consequently the part of

$$(\varrho^\alpha+\varrho^{\alpha-1}\theta_1(\pi)+\varrho^{\alpha-2}\theta_2(\pi)+\ldots)^n,$$

which is independent of (π), is

$$\varrho^{\alpha n}+\Sigma\theta_1(\alpha n)\varrho^{\alpha n-1}+(\Sigma\theta_1(\alpha n-1)\Sigma\theta_1(\alpha n)+\Sigma\theta_2(\alpha n))\varrho^{\alpha n-2}$$
$$+(\Sigma\theta_1(\alpha n-2)\Sigma\theta_1(\alpha n-1)\Sigma\theta_1(\alpha n)+\Sigma\theta_1(\alpha n-2)\Sigma\theta_2(\alpha n)$$
$$+\Sigma\theta_2(\alpha n-1)\Sigma\theta_1(\alpha n)+\Sigma\theta_3(\alpha n))\varrho^{\alpha n-3}+\ldots$$

SECTION III. *On the Solution of Linear Differential Equations with Variable Coefficients.*

The general linear differential equation

$$\mathrm{X}_r\frac{d^ru}{dx^r}+\mathrm{X}_{r-1}\frac{d^{r-1}u}{dx_{r-1}}+\mathrm{X}_{r-2}\frac{d^{r-2}u}{dx^{r-2}}+\&c.=\mathrm{X},$$

where X_r, X_{r-1} are rational and entire functions of (x), may, as Professor BOOLE has shown, be always expressed in the symbolical form

$$\varrho^n\phi_n(\pi)u+\varrho^{n-1}\phi_{n-1}(\pi)u+\ldots\varrho\phi_1(\pi)u+\phi_0(\pi)u=\mathrm{X},$$

where

$$\varrho=x, \text{ and } \pi=x\frac{d}{dx},$$

and $\phi_n(\pi)$, $\phi_{n-1}(\pi)$, &c. are rational and entire functions of (π).

Suppose that by using the methods explained in this paper, we are able to reduce this equation to the form

$$(\varrho\psi_1^{(n)}(\pi)+\psi_0^{(n)}(\pi))(\varrho\psi_1^{(n-1)}(\pi)+\psi_0^{(n-1)}(\pi))\ldots(\varrho\psi_1(\pi)+\psi_0(\pi))u=\mathrm{X}\text{[*]}.$$

Assume

$$\begin{aligned}
\varrho\psi_1^{(n)}(\pi)u_{n-1}+\psi_0^{(n)}(\pi)u_{n-1}&=\mathrm{X}\\
\varrho\psi_1^{(n-1)}(\pi)u_{n-2}+\psi_0^{(n-1)}(\pi)u_{n-2}&=u_{n-1}\\
\varrho\psi_1^{(n-2)}(\pi)u_{n-3}+\psi_0^{(n-2)}(\pi)u_{n-3}&=u_{n-2}\\
\&c.\qquad\qquad&=\&c.\\
\varrho\psi_1(\pi)\quad u\quad+\psi_0\quad(\pi)u\quad&=u_1.
\end{aligned}$$

We thus reduce the proposed differential equation to forms already treated of by Professor BOOLE.

We may much simplify the process already explained for treating the symbolical quantity $\varrho^n\phi_n(\pi)+\&c.+\phi_0(\pi)$, by remarking that $\psi_1(\pi)$ must be sought among the divisors of $\phi_n(\pi)$, $\psi_0(\pi)$ among the divisors of $\phi_0(\pi)$; and we shall make use of this principle in the following application of the preceding theory to the solution of differential equations.

We shall denominate the equation deduced in the former part of this memoir,

$$\phi_0(\pi)-\frac{\psi_0(\pi)}{\psi_1(\pi-1)}\phi_1(\pi-1)+\frac{\psi_0(\pi)\psi_0(\pi-1)}{\psi_1(\pi-1)\psi_1(\pi-2)}\phi_2(\pi-2)-\&c.=0,$$

the criterion of the factor $\varrho\psi_1(\pi)+\psi_0(\pi)$.

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[*] It may be proper to remind the reader that $\psi_1^{(n)}(\pi)$, $\psi^{(n-1)}(\pi)$, &c. have no reference to the functions derived from $\psi_1\pi$ by differentiation.

To integrate the differential equation,

$$x^2(x+1)^3\frac{d^3u}{dx^3}+3x(x+1)^3\frac{d^2u}{dx^2}+(x^3+4x^2+3x)\frac{du}{dx}-(2x-3)u=\mathrm{X}.$$

The symbolical form of this equation is

$$\varrho^3\pi^3u+\varrho^2(3\pi^3+\pi-1)u+3\varrho(\pi^3+1)u+\pi(\pi^2-1)u=\mathrm{X}x.$$

The divisor of π^3 is π only, the divisors of $\pi(\pi^2-1)$ are $\pi-1$, π, $\pi+1$; hence putting $\psi_1(\pi)=\pi$, $\psi_0\pi=\pi-1$, we find the criterion of the symbolical quantity $\varrho\psi_1(\pi)+\psi_0(\pi)$ to become

$$\pi(\pi^2-1)-3\{(\pi-1)^3+1\}+\{3(\pi-2)^3+(\pi-2)-1\}-(\pi-3)^3=0,$$

an identical equation.

Hence $\varrho\pi+(\pi-1)$ is an internal factor of

$$\varrho^3\pi^3+\varrho^2(3\pi^3+\pi-1)+3\varrho(\pi^3+1)+\pi(\pi^2-1);$$

and the equation may be written, effecting the internal division,

$$\{\varrho^2(\pi-1)^2+\varrho(\pi+1)(2\pi-3)+\pi(\pi+1)\}(\varrho\pi+(\pi-1))u=\mathrm{X}x;$$

or if $\varrho\pi+(\pi-1)u=u_1$,

$$\{\varrho^2(\pi-1)^2+\varrho(\pi+1)(2\pi-3)+\pi(\pi+1)\}u_1=\mathrm{X}x.$$

The only divisor of $(\pi-1)^2$ is $\pi-1$, the divisors of $\pi(\pi+1)$ are π and $\pi+1$; and by trial it is found that the divisor $\varrho(\pi-1)+(\pi+1)$ satisfies the criterion, and is therefore an internal factor. Hence, effecting the internal division,

$$(\varrho(\pi-2)+\pi)(\varrho(\pi-1)+(\pi+1))u_1=\mathrm{X}x,$$

and the differential equation becomes

$$(\varrho(\pi-2)+\pi)(\varrho(\pi-1)+(\pi+1))(\varrho\pi+(\pi-1))u=\mathrm{X}x,$$

or

$$\left\{(x^2+x)\frac{d}{dx}-2x\right\}\left\{(x^2+x)\frac{d}{dx}-(x-1)\right\}\left\{(x^2+x)\frac{d}{dx}-1\right\}u=\mathrm{X}x.$$

Hence, performing the inverse calculations, we find for the complete integral;

$$u=\frac{x}{x+1}\int\frac{dx(x+1)^2}{x^3}\int\frac{dx}{x+1}\int\frac{\mathrm{X}dx}{(x+1)^3},$$

the three arbitrary constants being included under the signs of integration.

In case this method does not succeed, we may sometimes resolve the symbolical function into factors by assuming $u=(\pi+\xi)v$ and proceeding as before, determining (α) from the criterion, as will be shown in the following examples:—

To integrate the differential equation

$$x^2(x+1)^3\frac{d^2u}{du^2}+x(4x^3+11x^2+10x+3)\frac{du}{dx}+2x^3+10x^2+5x-3=\mathrm{X}.$$

The symbolical form of the equation is

$$\varrho^3(\pi^2+3\pi+2)+\varrho^2(3\pi^2+8\pi+10)+\varrho(3\pi^2+7\pi+5)+\pi^2+2\pi-3=\mathrm{X}.$$

Let $u=(\pi+\xi)v$, and the equation becomes

$$\varrho^3(\pi+1)(\pi+2)(\pi+\xi)v+\varrho^2(3\pi^2+8\pi+10)(\pi+\xi)v$$
$$+\varrho(3\pi^2+7\pi+5)(\pi+\xi)v+(\pi-1)(\pi+3)(\pi+\xi)v=\mathrm{X}.$$

Let $\psi_1(\pi)=\pi+2$, $\psi_0(\pi)=\pi+\xi$, then the criterion of $\varrho(\pi+2)+(\pi+\xi)$ become

$$(\pi^2+2\pi-3)(\pi+\xi)-\frac{\pi+\xi}{\pi+1}(3\pi^2+\pi+1)(\pi+\xi-1)$$
$$+\frac{(\pi+\xi)(\pi+\xi-1)}{(\pi+1)\pi}(3\pi^2-4\pi+6)(\pi+\xi-2)$$
$$-\frac{(\pi+\xi)(\pi+\xi-1)(\pi+\xi-2)}{(\pi+1)\pi(\pi-1)}(\pi^2-3\pi+2)(\pi+\xi-3)=0.$$

Put $\pi=0$ to determine ξ, and we have $\xi=0$ as one value of ξ, which on trial is found to satisfy the proposed.

Hence $\varrho(\pi+2)+\pi$ is an internal factor of the symbolical function

$$\varrho^3\pi(\pi+1)(\pi+2)+\varrho^2(3\pi^3+8\pi^2+10\pi)$$
$$+\varrho(3\pi^3+7\pi^2+5\pi)+\pi(\pi-1)(\pi+3).$$

Wherefore, effecting the internal division, the equation becomes

$$(\varrho^2\pi+\varrho(2\pi+3)+\pi+3)(\pi-1)(\varrho(\pi+2)+\pi)v=\mathrm{X},$$

whence performing the inverse calculations, we have

$$v=\frac{1}{(x+1)^2}\int dx(x+1)\int\frac{dx(x+1)^3}{x^5}\int\frac{dx.x^2.\mathrm{X}}{(x+1)^5};$$

$$\therefore\quad u=x\frac{d}{dx}\left\{\frac{1}{(x+1)^2}\int dx(x+1)\int\frac{dx(x+1)^3}{x^5}\int\frac{dx.x^2\mathrm{X}}{(x+1)^5}\right\},$$

where the arbitrary constants must be reduced to two.

Next consider the differential equation

$$(x^4+2x^3+x^2)\frac{d^2u}{dx^2}-6(x^2+x)\frac{du}{dx}+6(x+2)u=\mathrm{X};$$

the symbolical form of this equation is

$$\varrho^2\pi(\pi-1)u+2\varrho(\pi-1)(\pi-3)u+(\pi-3)(\pi-4)u=\mathrm{X}.$$

Let $u=(\pi+\xi)v$, and the equation becomes

$$\varrho^2\pi(\pi-1)(\pi+\xi)v+2\varrho(\pi-1)(\pi-3)(\pi+\xi)v+(\pi-3)(\pi-4)(\pi+\xi)v=\mathrm{X}.$$

Let $\psi_1(\pi)=\pi-1$, $\psi_0(\pi)=\pi-3$, and the criterion becomes

$$(\pi-3)(\pi-4)(\pi+\xi)-\frac{\pi-3}{\pi-2}\{2(\pi-2)(\pi-4)\}(\pi+\xi-1)$$
$$+\frac{(\pi-3)(\pi-4)}{(\pi-2)(\pi-3)}\{(\pi-2)(\pi-3)\}(\pi+\xi-2)=0.$$

Putting $\pi=0$ in this equation, we have $\xi=0$, and this value renders the above equation identical,

$$\therefore\quad \varrho(\pi-1)+(\pi-3)$$

is an internal factor of the symbolical function

$$\varrho^2\pi^2(\pi-1)+2\varrho\pi(\pi-1)(\pi-3)+\pi(\pi-3)(\pi-4);$$

wherefore, effecting the internal division, the equation becomes

$$\{\varrho(\pi-1)^2+\pi(\pi-4)\}\{\varrho(\pi-1)+(\pi-3)\}u=\mathrm{X}.$$

This equation may be written

$$\left\{\frac{(\pi-2)(\pi-3)(\pi-4)}{\pi-1}\right\}(\varrho+1)\left\{\frac{\pi(\pi-1)}{(\pi-2)(\pi-3)}\right\}(\varrho(\pi-1)+(\pi-3))=\mathrm{X},$$

in which the inverse calculations are all practicable.

As a final example we take the differential equation

$$(x^5+4x^4+5x^3+2x^2)\frac{d^2u}{dx^2}+(2x^4+3x^3+5x^2-6x)\frac{du}{dx}+(x+1)^2u=\mathrm{X}.$$

The symbolical form of this equation is

$$\varrho^3\pi(\pi+1)u+\varrho^2(4\pi^2-\pi+1)u+\varrho(5\pi^2-5\pi+2)u+(\pi-1)(2\pi-1)u=\mathrm{X}.$$

If we put $u=\pi v$, the equation becomes

$$(\varrho+1)(\pi-1)(\varrho(\pi-1)+\pi)(\varrho\pi+(2\pi-1))v=\mathrm{X},$$

in which the inverse calculations necessary for the solution of the equation are all practicable.

In cases where the assumption $u=(\pi+\xi)v$ does not lead to the solution of the equation, we may assume $u=(\pi+\xi_1)(\pi+\xi_2)v$, and proceed as before.

We may also treat linear differential equations by ascertaining the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of this symbolical expression,

$$\phi_n(\varrho).\pi^n+\phi_{n-1}(\varrho)\pi^{n-1}+\text{ \&c. }+\phi_1(\varrho)\pi+\phi_0(\varrho).$$

I have shown how this is to be effected when $n=2$ or 3.

For higher degrees the investigation would be very laborious. In all cases in which the second member of the differential equation is zero, this internal factor, supposing it to exist, would conduct us to a particular integral.